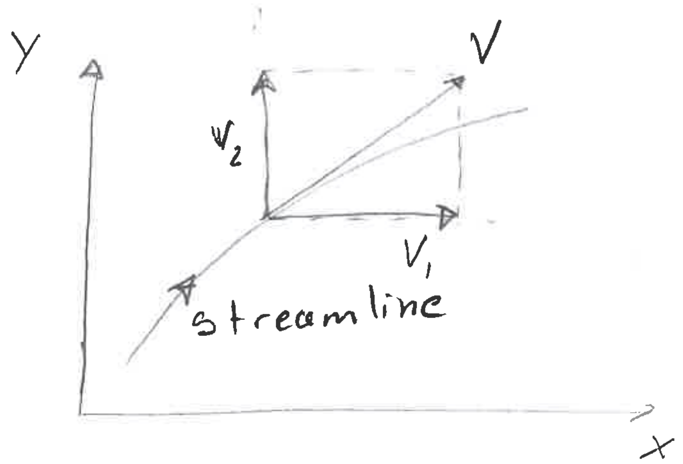
Fluid Flow (Potential theory)

Laplace's equation also plays a basic role in hydrodynamics, steady nonviscous fluid flow under physical conditions. Assuming two-dimensional analysis, so that the velocity vector (V) by which the motion of the fluid can be given depends only on two variables x and y (space variables)



Then we can use for V a complex function

$$V = V_1 + iV_2$$

Giving a magnitude $|V|$ and direction $\operatorname{Arg}(V)$ of velocity at each point $z = x + iy$. Here V_1 and V_2 are components of the velocity in the x and y directions. V is tangential to the path of the moving particles, called streamline of the motion.

For a given flow exits an analytic function

$F(z) = \phi(x,y) + i\psi(x,y)$, this is called

complex potential of the flow, such that

the streamline are given by $\psi(x,y) = \text{constant}$

and the velocity is given by

$$V = V_1 + iV_2 = \overline{F'(z)};$$

The bar denotes the complex conjugate.

ψ is the stream function. The function ϕ is

called the velocity potential. The curves $\phi = \text{constant}$

are called equipotential lines. V is the gradient

of ϕ; by definition,

$$V_1 = \frac{\partial \phi}{\partial x} \qquad V_2 = \frac{\partial \phi}{\partial y}$$ This is true

for the Cauchy-Riemann equation, F is

analytic in a domain D if only if the first

derivative satisfy the two

$$\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y} \quad \text{and} \quad \frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x}.$$

$$F'(z) = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} - i\frac{\partial \phi}{\partial y}$$

where $\dfrac{\partial \psi}{\partial x} = -\dfrac{\partial \phi}{\partial y}$

Example

The complex potential

$F(z) = z^2 = x^2 - y^2 + 2ixy$ describes a flow

with Equipotential $\phi = x^2 - y^2 = \text{constant}$

Streamlines $\psi = 2xy = \text{constant}$

the velocity vector is obtain as.

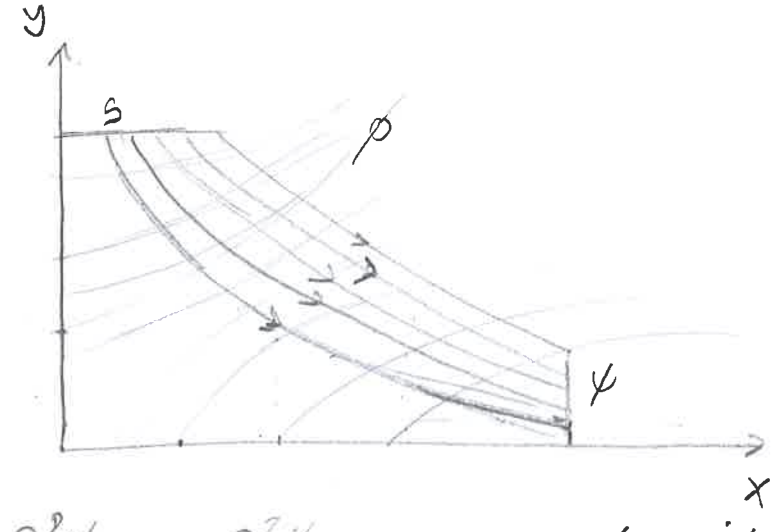
$\overline{F'(z)} = V$

$F'(z) = 2x + i2y$

$\overline{F'(z)} = 2x - i2y = V$

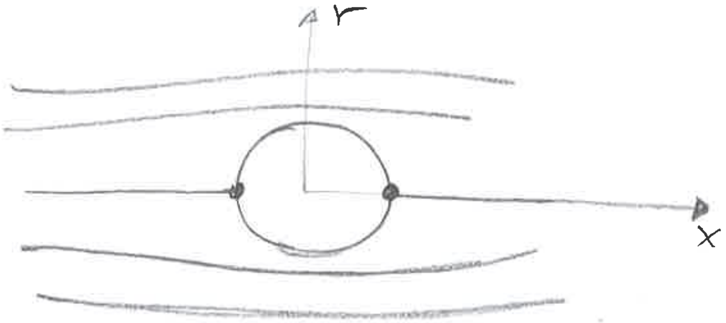
$V_1 = 2x$ and $V_2 = -2y$

The magnitude is $|V| = \sqrt{4x^2 + 4y^2} = 2\sqrt{x^2 + y^2}$



$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0 \quad \text{Continuity}$$

$2 - 2 = 0$ satisfyes the continuity equation



Special case.

$\psi(x,y)=0$

(1) $\left(r-\frac{1}{r}\right)=0$ or

(2) $\sin\theta=0$

$r=\pm 1$ we take the positive, the streamline is the circle of radius $r=1$ and x-axis ($\theta=0$ and $\theta=\pi$)

Stagnation points

$V=0$ for stagnation point

$$\left[F'(z)=1-\frac{1}{z^2}=0 \qquad z=\pm 1\right]$$